

TWISTORS OF ALMOST QUATERNIONIC MANIFOLDS

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ABSTRACT. We investigate the integrability of almost complex structures on the twistor space of an almost quaternionic manifold constructed with the help of a quaternionic connection. We show that if there is an integrable structure it is independent on the quaternionic connection. In dimension four, we express the anti-self-duality condition in terms of the Riemannian Ricci forms with respect to the associated quaternionic structure.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

An almost hyper-complex structure on a $4n$ -dimensional manifold M is a triple $H = (J_\alpha), \alpha = 1, 2, 3$, of almost complex structures $J_\alpha : TM \rightarrow TM$ satisfying the quaternionic identities $J_\alpha^2 = -id$ and $J_1 J_2 = -J_2 J_1 = J_3$. When each J_α is a complex structure, H is said to be a hyper complex structure on M .

An almost quaternionic structure on M is a rank-3 subbundle $\mathcal{Q} \subset \text{End}(TM)$ which is locally spanned by an almost hyper-complex structure $H = (J_\alpha)$. Such a locally defined triple H is called an admissible basis of \mathcal{Q} . A linear connection ∇ on TM is called a quaternionic connection if ∇ preserves \mathcal{Q} , i.e. $\nabla_X \sigma \in \Gamma(\mathcal{Q})$ for all vector fields X and smooth sections $\sigma \in \Gamma(\mathcal{Q})$. An almost quaternionic structure is said to be quaternionic if there is a torsion-free quaternionic connection. A \mathcal{Q} -hermitian metric is a Riemannian metric which is Hermitian

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with respect to each almost complex structure in \mathcal{Q} . An almost quaternionic (resp. quaternionic) manifold with \mathcal{Q} -hermitian metric is called an almost quaternionic Hermitian (resp. quaternionic hermitian) manifold

For $n = 1$ an almost quaternionic structure is the same as an oriented conformal structure and it turns out to be always quaternionic. The existence of a (local) hyper-complex structure is a strong condition since the integrability of the (local) almost hyper-complex structure implies that the corresponding conformal structure is anti-self-dual [6].

When $n \geq 2$, the existence of torsion-free quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, \mathbf{H})Sp(1)$ structure [8, 22, 26]. If the Levi-Civita connection ∇^g of a quaternionic hermitian manifold (M, g, \mathcal{Q}) is a quaternionic connection then (M, g, \mathcal{Q}) is called a Quaternionic Kähler manifold (briefly QK manifold). This condition is equivalent to the statement that the holonomy group of g is contained in $Sp(n)Sp(1)$ [1, 2, 23, 25, 17]. If on a QK manifold there exists an admissible basis (H) such that each almost complex structure $(J_\alpha) \in (H), \alpha = 1, 2, 3$ is parallel with respect to the Levi-Civita connection then the manifold is called hyperKähler (briefly HK). In this case the holonomy group of g is contained in $Sp(n)$.

The various notions of quaternionic manifolds arise in a natural way from the theory of supersymmetric sigma models as well as in string theory. The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion [13, 14]. The geometry of (4,0) supersymmetric two-dimensional sigma models without Wess-Zumino term (torsion) is a hyperKähler manifold. In the presence of torsion the geometry of the target space becomes hyperKähler with torsion (briefly HKT) [15]. This means that the complex structures $J_\alpha, \alpha = 1, 2, 3$, are parallel with respect to a metric quaternionic connection with totally skew-symmetric torsion [15]. Local (4,0) supersymmetry requires that the target space of two dimensional sigma models with Wess-Zumino term be either HKT or quaternionic Kähler with torsion (briefly QKT) [21] which means that the quaternionic subbundle is parallel with respect to a metric linear connection with totally skew-symmetric torsion and the torsion 3-form is of type $(1,2)+(2,1)$ with respect to all almost complex structures in \mathcal{Q} . The target space of two-dimensional (4,0) supersymmetric sigma models with torsion coupled to (4,0) supergravity is a QKT manifold [16].

The main object of interest in this article are properties of the twistor space of an almost quaternionic manifold. The twistor space Z is the unit sphere bundle with fibre S^2 consisting of all almost complex structures compatible with the given almost quaternionic structure \mathcal{Q} . We consider two almost complex structures I_1^∇, I_2^∇ on the twistor space Z over an almost quaternionic manifold generated by a quaternionic connection ∇ . The structure $I_1^{\nabla^g}$ was originally constructed for four-dimensional Riemannian manifold in [6], $I_2^{\nabla^g}$ is described in [11]. In both cases the horizontal space of the Levi-Civita connection is used. It is shown in [6] that $I_1^{\nabla^g}$ is integrable exactly when the conformal structure is anti-self-dual while in [11] the non-integrability of $I_2^{\nabla^g}$ is proved. The integrability of $I_1^{\nabla^g}$ for quaternionic Kähler space is established in [23] and then it is generalized for quaternionic manifold in [24]. For HKT and QKT spaces the integrability of I_1^∇ is proved in [15] and [16], respectively. Hermitian and Kähler geometry of the twistor space is investigated in [12, 23]. The almost hermitian geometry of the twistor space over a 4-dimensional Riemannian manifold is studied in [10].

The almost hermitian geometry of the twistor space over a quaternionic Kähler manifold, HKT and QKT spaces is considered in [3, 19].

In the present note we investigate the dependence on the quaternionic connection of these naturally defined almost complex structures on the twistor space over an almost quaternionic manifold. We obtain conditions on the quaternionic connection which imply the coincidence of the corresponding almost complex structures (Corollary 3.3, Corollary 3.2). We show that the existence of an integrable almost complex structure on the twistor space does not depend on the quaternionic connection and it is equivalent to the condition that the almost quaternionic $4n$ -manifold is quaternionic for $n \geq 2$ (Theorem 3.7, Theorem 3.10). In particular, we show that the integrable almost complex structures $I_1^{\nabla^g}$ and I_1^{∇} on a QKT space coincide.

In dimension four we find new relations between the Riemannian Ricci forms, i.e. the 2-forms which determine the $Sp(1)$ -component of the Riemannian curvature, which are equivalent to the anti-self-duality of the oriented conformal structure corresponding to a given quaternionic structure (Theorem 3.6).

2. PRELIMINARIES

Let \mathbf{H} be the quaternions and identify $\mathbf{H}^n = \mathbf{R}^{4n}$. To fix notation we assume that \mathbf{H} acts on \mathbf{H}^n by right multiplication. This defines an antihomomorphism

$$\begin{aligned} \lambda : \{\text{unit quaternions}\} &= \\ &= \{x + j_1 y + j_2 z + j_3 w \mid x^2 + y^2 + z^2 + w^2 = 1\} \longrightarrow SO(4n) \subset GL(4n, \mathbf{R}), \end{aligned}$$

where our convention is that $SO(4n)$ acts on \mathbf{H}^n on the left. Denote the image by $Sp(1)$ and let $J_1^0 = -\lambda(j_1)$, $J_2^0 = -\lambda(j_2)$, $J_3^0 = -\lambda(j_3)$. The Lie algebra of $Sp(1)$ is $sp(1) = \text{span}\{J_1^0, J_2^0, J_3^0\}$ and we have

$$J_1^{02} = J_2^{02} = J_3^{02} = -1, \quad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0.$$

Define $GL(n, H) = \{A \in GL(4n, \mathbf{R}) : A(sp(1))A^{-1} = sp(1)\}$. The Lie algebra of $GL(n, H)$ is $gl(n, H) = \{A \in gl(4n, \mathbf{R}) : AB = BA \text{ for all } B \in sp(1)\}$.

Let (M, \mathcal{Q}) be an almost quaternionic manifold and $H = (J_a)$, $a = 1, 2, 3$ be an admissible local basis. Let $B \in \Lambda^2(TM)$. We say that B is of type $(0, 2)_{J_a}$ with respect to J_a if

$$B(J_a X, Y) = -J_a B(X, Y).$$

We denote this space by $\Lambda_{J_a}^{0,2}$. The projection $B_{J_a}^{0,2}$ is given by

$$B_{J_a}^{0,2}(X, Y) = \frac{1}{4} (B(J_a X, J_a Y) - B(X, Y) - J_a B(J_a X, Y) - J_a B(X, J_a Y)).$$

For example, the Nijenhuis tensor $N_a \in \Lambda_{J_a}^{0,2}$.

We denote the space of quaternionic connections on an almost quaternionic manifold by $\Delta(\mathcal{Q})$.

Let $\nabla \in \Delta(\mathcal{Q})$ be a quaternionic connection on an almost quaternionic manifold (M, \mathcal{Q}) . This means that there exist locally defined 1-forms ω_α , $\alpha = 1, 2, 3$ such that

$$(2.1) \quad \nabla J_a = -\omega_b \otimes J_c + \omega_c \otimes J_b.$$

Here and further (a, b, c) stands for a cyclic permutation of $(1, 2, 3)$.

It follows from (2.1) that the curvature R^∇ of any quaternionic connection $\nabla \in \Delta(\mathcal{Q})$ satisfies the relations

$$(2.2) \quad [R^\nabla, J_a] = -A_b \otimes J_c + A_c \otimes J_b, \quad A_a = d\omega_a + \omega_b \wedge \omega_c.$$

The Ricci 2-forms of a quaternionic connection are defined by

$$\rho_a^\nabla(X, Y) = -\frac{1}{2} \text{Tr} (Z \longrightarrow J_a R^\nabla(X, Y) Z), \quad a = 1, 2, 3.$$

It is easy to see, using (2.2), that the Ricci forms are given by

$$\rho_a^\nabla = d\omega_a + \omega_b \wedge \omega_c.$$

We split the curvature of ∇ into $gl(n, H)$ -valued part $(R^\nabla)'$ and $sp(1)$ -valued part $(R^\nabla)''$ following the classical scheme (see e.g. [4, 17, 7])

Proposition 2.1. *The curvature of an almost quaternionic connection on M splits as follows*

$$\begin{aligned} R^\nabla(X, Y) &= (R^\nabla)'(X, Y) + \frac{1}{2n}(\rho_1^\nabla(X, Y)J_1 + \rho_2^\nabla(X, Y)J_2 + \rho_3^\nabla(X, Y)J_3), \\ [(R^\nabla)'(X, Y), J_a] &= 0, \quad a = 1, 2, 3. \end{aligned}$$

Let Ω, Θ be the curvature 2-form and the torsion 2-form of ∇ on the principal $GL(n, H)Sp(1)$ -bundle $\mathcal{Q}(M)$, respectively ([20]). We denote the splitting of the $gl(n, H) \oplus sp(1)$ -valued curvature 2-form Ω on $\mathcal{Q}(M)$ according to Proposition 2.1, by $\Omega = \Omega' + \Omega''$, where Ω' is a $gl(n, H)$ -valued 2-form and Ω'' is a $sp(1)$ -valued form. Explicitly,

$$\Omega'' = \Omega_1'' J_1^0 + \Omega_2'' J_2^0 + \Omega_3'' J_3^0,$$

where $\Omega_a'', a = 1, 2, 3$, are 2-forms. If $\xi, \eta, \zeta \in \mathbf{R}^{4n}$, then the 2-forms $\Omega_a'', a = 1, 2, 3$, are given by

$$(2.3) \quad \Omega_a''(B(\xi), B(\eta)) = \frac{1}{2n} \rho_a(X, Y), \quad X = u(\xi), Y = u(\eta).$$

3. TWISTOR SPACE OF ALMOST QUATERNIONIC MANIFOLDS

In this section we adapt the setup from [27, 9] to incorporate a torsion. Our discussion is very close to that of [3, 19].

Let M be a $4n$ -dimensional manifold endowed with an almost quaternionic structure \mathcal{Q} . Let J_1, J_2, J_3 be an admissible basis of \mathcal{Q} defined in some neighborhood of a given point $p \in M$. Any linear frame u of $T_p M$ can be considered as an isomorphism $u : \mathbf{R}^{4n} \longrightarrow T_p M$. If we pick such a frame u we can define a subspace of the space of the all endomorphisms of $T_p M$ by $u(sp(1))u^{-1}$. Clearly, this subset is a quaternionic structure at the point p and in the general case this quaternionic structure is different from \mathcal{Q}_p . We define $\mathcal{Q}(M)$ to be the set of all linear frames u which satisfy $u(sp(1))u^{-1} = \mathcal{Q}$. It is easy to see that $\mathcal{Q}(M)$ is a principal frame bundle of M with structure group $GL(n, H)Sp(1)$, it is also called a $GL(n, H)Sp(1)$ -structure on M .

Let $\pi : \mathcal{Q}(M) \longrightarrow M$ be the natural projection. For each $u \in \mathcal{Q}(M)$ we consider the linear isomorphisms $j(u)$ on $T_{\pi(u)} M$ defined by $j(u) = u J_3^0 u^{-1}$. It is easy to see that $(j(u))^2 = -id$. For each point $p \in M$ we define $Z_p(M) = \{j(u) : u \in \mathcal{Q}(M), \pi(u) = p\}$. In other words, $Z_p(M)$ is the space of all complex structures in the tangent space $T_p M$ which are compatible with the almost quaternionic structure on M .

We define the twistor space Z of (M, \mathcal{Q}) by setting $Z = \bigcup_{p \in M} Z_p(M)$. Let H_3 be the stabilizer of J_3^0 in the group $GL(n, H)Sp(1)$. There is a bijective correspondence between the symmetric space $GL(n, H)Sp(1)/H_3 \cong S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and $Z_p(M)$ for each $p \in M$. So we can consider Z as the associated fibre bundle of $\mathcal{Q}(M)$ with standard fibre $GL(n, H)Sp(1, \mathbf{R})/H_3$. Hence, $\mathcal{Q}(M)$ is a principal fibre bundle over Z with structure group H_3 and projection j . We consider the symmetric spaces $GL(n, H)Sp(1)/H_3$. We have the following Cartan decomposition $gl(n, H) \oplus sp(1) = h_3 \oplus m_3$ where

$$h_3 = \{A \in gl(n, H) \oplus sp(1) : AJ_3^0 = J_3^0 A\}$$

is the Lie algebra of H_3 and $m_3 = \{A \in gl(n, H) \oplus sp(1) : AJ_3^0 = -J_3^0 A\}$. It is clear that m_3 is generated by J_1^0, J_2^0 , i.e. $m_3 = \text{span}\{J_1^0, J_2^0\}$. Hence, if $A \in m_3$ then $J_3^0 A \in m_3$.

Let ∇ be a quaternionic connection on M , i.e. ∇ is a linear connection in the principal bundle $\mathcal{Q}(M)$ (see e.g. [20]). Note that we make no assumptions on the torsion or on the curvature of ∇ . Keeping in mind the similarity with the 4-dimensional and quaternionic geometry [6, 11, 23, 24], we use ∇ to define two almost complex structures I_1^∇ and I_2^∇ on the twistor space Z . The construction of these almost complex structures depends on the choice of the quaternionic connection ∇ .

We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on $\mathcal{Q}(M)$ corresponding to $A \in gl(n, H) \oplus sp(1)$ (resp. $\xi \in \mathbf{R}^{4n}$).

Let $u \in \mathcal{Q}(M)$ and Q_u be the horizontal subspace of the tangent space $T_u \mathcal{Q}(M)$ induced by ∇ (see e.g. [20]). The vertical space i.e. the vector space tangent to a fibre, is isomorphic to

$$(gl(n, H) \oplus sp(1))_u^* = (h_3)_u^* \oplus (m_3)_u^*,$$

where $(h_3)_u^* = \{A_u^* : A \in h_3\}$, $(m_3)_u^* = \{A_u^* : A \in m_3\}$.

Hence, $T_u \mathcal{Q}(M) = (h_3)_u^* \oplus (m_3)_u^* \oplus Q_u$.

For each $u \in \mathcal{Q}(M)$, we put

$$V_{j(u)} = j_{*u}((m_3)_u^*), \quad H_{j(u)} = j_{*u}Q_u.$$

Thus we obtain vertical and horizontal distributions V and H on Z . Since $\mathcal{Q}(M)$ is a principal fibre bundle over Z with structure group H_3 we have $\text{Ker } j_{*u} = (h_3)_u^*$.

Hence $V_{j(u)} = j_{*u}(m_3)_u^*$ and $j_{*u}|_{(m_3)_u^* \oplus Q_u} : (m_3)_u^* \oplus Q_u \longrightarrow T_{j(u)}Z$ is an isomorphism.

We define two almost complex structures I_1^∇ and I_2^∇ on Z by

$$(3.4) \quad \begin{aligned} I_1^\nabla(j_{*u}A^*) &= j_{*u}(J_3^0 A)^*, & I_2^\nabla(j_{*u}A^*) &= -j_{*u}(J_3^0 A)^* \\ I_i^\nabla(j_{*u}B(\xi)) &= j_{*u}B(J_3^0 \xi), & i &= 1, 2, \end{aligned}$$

for $A \in m_3, \xi \in \mathbf{R}^{4n}$.

For twistor bundles of 4-dimensional Riemannian manifolds the almost complex structure $I_1^{\nabla^g}$ is introduced in [6] and the almost complex structure $I_2^{\nabla^g}$ is introduced in [11] in terms of the horizontal spaces of the Levi-Civita connection. It is well known that $I_1^{\nabla^g}$ is integrable exactly when the 4-manifold is anti-self-dual [6], while $I_2^{\nabla^g}$ is never integrable [11]. The almost complex structure $I_1^{\nabla^g}$ and its integrability for QK spaces is established in [23] and then generalized for quaternionic manifold in [24]. The integrability of I_1^∇ in the case of HKT and QKT manifolds is established in [15, 16], respectively. The non-integrability of $I_2^{\nabla^g}$ in the case of QK is confirmed in [10] and the non-integrability of I_1^∇ for QKT spaces is done in [19].

3.1. Dependence on the quaternionic connection. In this section we investigate when different almost quaternionic connections induce the same almost complex structure on the twistor space over an almost quaternionic manifold.

Let ∇ and ∇' be two different quaternionic connections on an almost quaternionic manifold (M, \mathcal{Q}) . Then we have

$$\nabla'_X = \nabla_X + S_X, \quad X \in \Gamma(TM),$$

where S_X is a $(1,1)$ tensor on M and $u^{-1}(S_X)u$ belongs to $gl(n, H) \oplus sp(1)$ for any $u \in \mathcal{Q}(M)$. Thus we have the splitting

$$(3.5) \quad S_X(Y) = S_X^0(Y) + s^1(X)J_1Y + s^2(X)J_2Y + s^3(X)J_3Y,$$

where $X, Y \in \Gamma(TM)$, s^i are 1-forms and $[S_X^0, J_i] = 0$, $i = 1, 2, 3$.

Proposition 3.1. *Let ∇ and ∇' be two different quaternionic connections on an almost quaternionic manifold (M, \mathcal{Q}) . The following conditions are equivalent:*

- i). *The two almost complex structures I_1^∇ and $I_1^{\nabla'}$ on the twistor space Z coincide.*
- ii). *The 1-forms s^1, s^2, s^3 are related as follows*

$$s^1(J_1X) = s^2(J_2X) = s^3(J_3X), \quad X \in \Gamma(TM).$$

Proof. We fix a point J of the twistor space Z . We have $J = a_1J_1 + a_2J_2 + a_3J_3$ with $a_1^2 + a_2^2 + a_3^2 = 1$. Let $\pi : Z \rightarrow M$ be the natural projection and $x = \pi(J)$. The connection ∇ induces a splitting of the tangent space of Z into vertical and horizontal components: $T_JZ = V_J \oplus H_J$. Let v and h be the vertical and horizontal projections corresponding to this splitting. Let $T_JZ = V'_J \oplus H'_J$ be the splitting induced by ∇' with the projections v' and h' , respectively. It is easy to observe the following identities

$$(3.6) \quad \begin{aligned} v + h &= 1 \\ v' + h' &= 1 \\ vv' &= v' \\ v' + vh' &= v \end{aligned}$$

In fact, $V_J = V'_J$ and we may regard this space as a subspace of \mathcal{Q}_x . We have that

$$V_J = \{W \in \mathcal{Q}_x \mid WJ + JW = 0\} = \{w_1J_1 + w_2J_2 + w_3J_3 \mid w_1a_1 + w_2a_2 + w_3a_3 = 0\},$$

where $J = a_1J_1 + a_2J_2 + a_3J_3$. It follows that for any $W \in V_J$, $I_1^\nabla(W) = I_1^{\nabla'}(W) = JW$. In general, for any $W \in T_JZ$, we have

$$(3.7) \quad \begin{aligned} I_1^\nabla(W) &= J(vW) + (J\pi(W))^h \\ I_1^{\nabla'}(W) &= J(v'W) + (J\pi(W))^{h'}, \end{aligned}$$

where $(\cdot)^h$ (resp. $(\cdot)^{h'}$) denotes the horizontal lift on Z of the corresponding vector field on M with respect to ∇ (resp. ∇'). Using (3.6), we calculate that

$$(3.8) \quad \begin{aligned} v(I_1^{\nabla'}W) &= J(v'W) + v(J\pi(W))^{h'} = J((v - vh')W) + v(J\pi(W))^{h'} = \\ &= v(I_1^\nabla W) - J(vh'W) + v(J\pi(W))^{h'}. \end{aligned}$$

We investigate the equality

$$(3.9) \quad J(vh'W) = v(J\pi(W))^{h'}, \quad W \in T_J Z.$$

Take $W = Y^{h'}$, $Y \in \Gamma(TM)$ in (3.9) to get

$$(3.10) \quad J(vY^{h'}) = v(JY)^{h'}, \quad Y \in T_x M$$

Hence, (3.10) is equivalent to $I_1^\nabla = I_1^{\nabla'}$ because of (3.8).

Let (U, x_1, \dots, x_{4n}) be a local coordinate system on M and let $Y = \sum Y^i \frac{\partial}{\partial x^i}$. The horizontal lift of Y with respect to ∇' at the point $J \in Z$ is given by

$$(3.11) \quad Y_J^{h'} = \sum_{i=1}^{4n} (Y^i \circ \pi) \frac{\partial}{\partial x^i} - \sum_{s=1}^3 a_s \nabla'_Y J_s$$

We calculate

$$(3.12) \quad \begin{aligned} v(JY)^{h'} &= (JY)^{h'} - h(JY)^{h'} = (JY)^{h'} - (JY)^h = \\ &= \sum_{s=1}^3 a_s (-\nabla'_{JY} J_s + \nabla_{JY} J_s) = -[S_{JY}, J] \end{aligned}$$

On the other hand, we have

$$(3.13) \quad J(vY^{h'}) = J(Y^{h'} - Y^h) = J \sum_{s=1}^3 a_s (-\nabla'_Y J_s + \nabla_Y J_s) = -J[S_Y, J]$$

Substitute (3.12) and (3.13) into (3.10) to get that $I_1^\nabla = I_1^{\nabla'}$ is equivalent to the condition

$$(3.14) \quad J[S_Y, J] = [S_{JY}, J], \quad Y \in \Gamma(TM), J \in Z.$$

Now, (3.14) easily leads to the equivalence of i) and ii). \square

We note that (3.14) is discovered in connection with the coincidence of the almost complex structures generated by two Oproiu connections in [5].

Corollary 3.2. *Let ∇ and ∇' be two different quaternionic connections on an almost quaternionic manifold (M, \mathcal{Q}) . The following conditions are equivalent:*

- i). *The two almost complex structures I_2^∇ and $I_2^{\nabla'}$ on the twistor space Z coincide.*
- ii). *The 1-forms s^1, s^2, s^3 vanish identically, $s_1 = s_2 = s_3 = 0$.*

Proof. It is sufficient to observe from the proof of Proposition 3.1 that $I_2^\nabla = I_2^{\nabla'}$ is equivalent to $J[S_Y, J] = -[S_{JY}, J]$, $Y \in \Gamma(TM), J \in Z$. The latter condition implies $s_1 = s_2 = s_3 = 0$. \square

Corollary 3.3. *Let ∇ and ∇' be two different quaternionic connections with torsion tensors $T^{\nabla'}$ and T^∇ , respectively, on an almost quaternionic manifold (M, \mathcal{Q}) . The following conditions are equivalent:*

- i). *The two almost complex structures I_1^∇ and $I_1^{\nabla'}$ on the twistor space Z coincide.*

ii). The $(0, 2)_J$ part with respect to all $J \in \mathcal{Q}$ of the torsion T^∇ and $T^{\nabla'}$ coincides,

$$(T^\nabla)_J^{0,2} = (T^{\nabla'})_J^{0,2}.$$

Proof. Let $S = \nabla' - \nabla$. Then we have

$$(3.15) \quad T^{\nabla'}(X, Y) = T^\nabla(X, Y) + S_X(Y) - S_Y(X).$$

The $(0, 2)_J$ -part with respect to J of (3.15) gives

$$(3.16) \quad (T^{\nabla'})_J^{0,2} - (T^\nabla)_J^{0,2} = [S_{JX}, J]Y - J[S_X, J]Y - [S_{JY}, J]X + J[S_Y, J]X.$$

For example, put $J = J_3$ in (3.16) and use the splitting (3.5) to obtain

$$(3.17) \quad (T^{\nabla'})_{J_3}^{0,2} - (T^\nabla)_{J_3}^{0,2} = (s_1(J_3X) - s_2(X))J_2Y - (s_1(X) + s_2(J_3X))J_1Y \\ - (s_1(J_3Y) - s_2(Y))J_2X + (s_1(Y) + s_2(J_3Y))J_1X.$$

Hence we get $s_1(J_1X) = s_2(J_2X)$ is equivalent to $(T^{\nabla'})_{J_3}^{0,2} = (T^\nabla)_{J_3}^{0,2}$.

Similarly, put $J = J_1$ in (3.16) and using (3.5) one gets $s_3(J_3X) = s_2(J_2X)$ is equivalent to $(T^{\nabla'})_{J_1}^{0,2} = (T^\nabla)_{J_1}^{0,2}$. Hence, $s_1(J_1X) = s_2(J_2X) = s_3(J_3X)$ is equivalent to $(T^{\nabla'})_{J_3}^{0,2} = (T^\nabla)_{J_3}^{0,2}$ and $(T^{\nabla'})_{J_1}^{0,2} = (T^\nabla)_{J_1}^{0,2}$. The latter conditions imply $(T^{\nabla'})_{J_2}^{0,2} = (T^\nabla)_{J_2}^{0,2}$ since the formula (3.4.4) in [4] expressing N_{J_3} by N_{J_1} and N_{J_2} holds for the $(0, 2)_{J_a}$ -part $T_{J_a}^\nabla, a = 1, 2, 3$, of the torsion. It is easy to see that the general formula (6) in [5] which expresses N_J in terms of $N_{J_1}, N_{J_2}, N_{J_3}$ holds also for the $(0, 2)_J$ -part of any tensor from $\Lambda^2(TM)$. Applying this formula to the $(0, 2)_J$ -part of the torsion, we conclude that $(T^{\nabla'})_J^{0,2} = (T^\nabla)_J^{0,2}$ is equivalent to $s_1(J_1X) = s_2(J_2X) = s_3(J_3X)$. Proposition 3.1 completes now the proof. \square

3.2. Integrability. In this section we investigate conditions on the almost quaternionic connection ∇ which imply the integrability of the almost complex structure I_1^∇ on Z . We also show that I_2^∇ is never integrable i.e. for any choice of the almost quaternionic connection ∇ it has non-vanishing Nijenhuis tensor.

We denote by $IN_i, i = 1, 2$ the Nijenhuis tensors of I_i and recall that

$$IN_i(U, W) = [I_iU, I_iW] - [U, W] - I_i[I_iU, W] - I_i[U, I_iW], \quad U, W \in \Gamma(TZ).$$

Proposition 3.4. *Let ∇ be a quaternionic connection on an almost quaternionic manifold (M, \mathcal{Q}) with torsion tensor T^∇ . The following conditions are equivalent:*

- i). *The almost complex structure I_1^∇ on the twistor space Z of (M, \mathcal{Q}) is integrable.*
- ii). *The $(0, 2)_J$ -part $(T^\nabla)_J^{0,2}$ of the torsion with respect to all $J \in \mathcal{Q}$ vanishes, and the $(2, 0) + (0, 2)$ parts of the Ricci 2-forms with respect to an admissible basis J_1, J_2, J_3 of \mathcal{Q} coincide, i.e. the next conditions hold*

$$(3.18) \quad (T^\nabla)_J^{0,2} = 0, J \in \mathcal{Q},$$

$$(3.19) \quad \rho_a(J_cX, J_cY) - \rho_a(X, Y) + \rho_b(J_cX, Y) + \rho_b(X, J_cY) = 0.$$

Proof. Let J_1, J_2, J_3 be an admissible basis of the almost quaternionic structure \mathcal{Q} .

Let hor be the natural projection $T_u Q \longrightarrow (m_3)_u^* \oplus Q_u$, with $\ker(hor) = (h_3)_u^*$. We define a tensor field I'_1 on $\mathcal{Q}(M)$ by

$$\begin{aligned} I'_1(U) &\in (m_3)_u^* \oplus Q_u, \\ j_{*u}(I'_1(U)) &= I_1(j_{*u}U), \quad U \in T_u P. \end{aligned}$$

For any $U, W \in \Gamma(T\mathcal{Q}(M))$ we define

$$IN'_1(U, W) = hor[I'_1U, I'_1W] - hor[horU, horW] - I'_1[horU, horW] - I'_1[horU, I'_1W]$$

It is easy to check that IN'_1 is a tensor field on $\mathcal{Q}(M)$. We also observe that

$$(3.20) \quad j_{*u}(IN'_1(U, W)) = IN_1(j_{*u}U, j_{*u}W), \quad U, W \in T_u \mathcal{Q}(M)$$

Let $A, B \in m_3$ and $\xi, \eta \in \mathbf{R}^{4n}$. Using the well known general commutation relations among the fundamental vector fields and standard horizontal vector fields on the principal bundle $\mathcal{Q}(M)$ (see e.g. [20]), we calculate taking into account (3.20) that

$$\begin{aligned} (3.21) \quad & IN_1(j_{*u}(A_u^*), j_{*u}(B_u^*)) = 0. \\ & IN_1(j_{*u}(A_u^*), j_{*u}(B(\xi)_u)) = 0. \\ & [IN_1(j_{*u}(B(\xi)_u), j_{*u}(B(\eta)_u))]_H = \\ & \quad j_{*u}(B(-\Theta(B(J_3^0\xi), B(J_3^0\eta)) + \Theta(B(\xi), B(\eta)) \\ & \quad + J_3^0\Theta(B(J_3^0\xi), B(\eta)) + J_3^0\Theta(B(\xi), B(J_3^0\eta)))_u). \end{aligned}$$

$$\begin{aligned} (3.22) \quad & [IN_1(j_{*u}(B(\xi)_u), j_{*u}(B(\eta)_u))]_V = \\ & \quad \{-\rho_1(B(J_3^0\xi), B(J_3^0\eta)) + \rho_1(B(\xi), B(\eta)) \\ & \quad - \rho_2(B(J_3^0\xi), B(\eta)) - \rho_2(B(\xi), B(J_3^0\eta))\}j_{*u}(J_1^0) \\ & \quad + \{-\rho_2(B(J_3^0\xi), B(J_3^0\eta)) + \rho_2(B(\xi), B(\eta)) \\ & \quad + \rho_1(B(J_3^0\xi), B(\eta)) + \rho_1(B(\xi), B(J_3^0\eta))\}j_{*u}(J_2^0). \end{aligned}$$

$$(3.23) \quad IN_2(j_{*u}(A_u^*), j_{*u}(B(\xi)_u)) = -4j_{*u}(B(A\xi)_u) \neq 0.$$

Take $X = u(\xi), Y = u(\eta)$, we see that (3.21) and (3.22) are equivalent to

$$(3.24) \quad (T^\nabla)_{J_3}^{0,2} = T^\nabla(J_3X, J_3Y) - T^\nabla(X, Y) - J_3T^\nabla(J_3X, Y) - J_3T^\nabla(X, J_3Y) = 0,$$

$$(3.25) \quad (\Pi^\nabla)_{J_3}^{0,2} = \rho_1^\nabla(J_3X, J_3Y) - \rho_1^\nabla(X, Y) + \rho_2^\nabla(J_3X, Y) + \rho_2^\nabla(X, J_3Y) = 0,$$

respectively.

Similarly, we get $(\Pi^\nabla)_{J_3}^{0,2} = (\Pi^\nabla)_{J_2}^{0,2} = (\Pi^\nabla)_{J_1}^{0,2} = 0$ and $(T^\nabla)_{J_3}^{0,2} = (T^\nabla)_{J_2}^{0,2} = (T^\nabla)_{J_1}^{0,2} = 0$. The first equalities imply that (3.22) is equivalent to (3.19). We apply to the second equalities the same arguments as in the proof of Corollary 3.3, i.e. formula (6) in [4], to derive that (3.21) is equivalent to $(T^\nabla)_J^{0,2} = 0$ for all local $J \in \mathcal{Q}$. \square

The equation (3.23) in the proof of Proposition 3.4 yield

Corollary 3.5. *Let ∇ be an almost quaternionic connection on an almost quaternionic manifold (M, \mathcal{Q}) with torsion tensor T^∇ . Then the almost complex structure I_2^∇ on the twistor space Z of (M, \mathcal{Q}) is never integrable.*

In the 4-dimensional case we derive

Theorem 3.6. *Let (M^4, g) be a 4-dimensional Riemannian manifold with a Riemannian metric g and let \mathcal{Q} be the quaternionic structure corresponding to the conformal class generated by g with a local basis J_1, J_2, J_3 . Then the following conditions are equivalent*

- i). *The metric g is anti-self-dual.*
- ii). *The Ricci forms ρ_a^g of the Levi-Civita connection ∇^g satisfy (3.19), i.e.*

$$\rho_a^g(J_c X, J_c Y) - \rho_a^g(X, Y) + \rho_b^g(J_c X, Y) + \rho_b^g(X, J_c Y) = 0.$$

- iii). *The torsion condition (3.18) for a linear connection ∇ always implies the curvature condition (3.19).*

Proof. The proof is a direct consequence of Proposition 3.4, Corollary 3.3 and the result in [6] which states that the almost complex structure $I_1^{\nabla^g}$ is integrable exactly when the conformal structure generated by g is anti-self-dual. \square

In higher dimensions, the curvature condition (3.19) is a consequence of the torsion condition (3.18) in the sense of the next

Theorem 3.7. *Let ∇ be a quaternionic connection on an almost quaternionic $4n$ -dimensional $n \geq 2$ manifold (M, \mathcal{Q}) with torsion tensor T^∇ . Then the following conditions are equivalent:*

- i). *The almost complex structure I_1^∇ on the twistor space Z of (M, \mathcal{Q}) is integrable.*
- ii). *The $(0, 2)_J$ -part $(T^\nabla)_J^{0,2}$ of the torsion with respect to all $J \in \mathcal{Q}$ vanishes,*

$$(T^\nabla)_J^{0,2} = 0, J \in \mathcal{Q}.$$

Proof. Suppose i) holds. Then ii) follows from Proposition 3.4.

For the converse, (3.18) and the fact that the connection ∇ is a quaternionic connection, $\nabla \in \Delta(\mathcal{Q})$, yield the next expression for the Nijenhuis tensor N_J of any local $J \in \mathcal{Q}$,

$$(3.26) \quad N_J(X, Y) \in \text{span}\{J_1 X, J_1 Y, J_2 X, J_2 Y, J_3 X, J_3 Y\},$$

where J_1, J_2, J_3 is an admissible local basis of \mathcal{Q} .

To prove that ii) implies the integrability of I_1^∇ , we apply the result in [4] (see also [5]) which states that an almost quaternionic $4n$ -manifold ($n \geq 2$) is quaternionic if and only if the three Nijenhuis tensors N_1, N_2, N_3 satisfy the condition

$$(3.27) \quad (N_1(X, Y) + N_2(X, Y) + N_3(X, Y)) \in \text{span}\{J_1 X, J_1 Y, J_2 X, J_2 Y, J_3 X, J_3 Y\}.$$

Clearly, (3.27) follows from (3.26) which shows that the almost quaternionic $4n$ -manifold ($n \geq 2$) (M, \mathcal{Q}) is a quaternionic manifold. Let ∇^0 be a torsion-free quaternionic connection on (M, \mathcal{Q}) . Then the almost complex structure $I_1^{\nabla^0}$ on the twistor space Z is integrable [24] and $I_1^\nabla = I_1^{\nabla^0}$ due to Corollary 3.3.

Hence, the equivalence between i) and ii) is established, which completes the proof. \square

From the proof of Proposition 3.4 and Theorem 3.7, we easily derive

Corollary 3.8. *Let ∇ be an almost quaternionic connection on an $4n$ -dimensional ($n \geq 2$) almost quaternionic manifold (M, \mathcal{Q}) with torsion tensor T^∇ . Then the torsion condition (3.18) implies the curvature condition (3.19).*

We note that Corollary 3.8 generalizes the same statement derived in the case of QKT-connection on QKT manifolds in [19].

Theorem 3.7 and Corollary 3.3 imply

Corollary 3.9. *Let (M, \mathcal{Q}) be an almost quaternionic manifold. Among the all almost complex structures $I_1^\nabla, \nabla \in \Delta(\mathcal{Q})$ on the twistor space Z at most one is integrable.*

The proof of the next theorem follows directly from the proof of Theorem 3.7, Theorem 3.6 and Corollary 3.9.

Theorem 3.10. *Let (M, \mathcal{Q}) be an almost quaternionic $4n$ -manifold. The next two conditions are equivalent:*

- 1). *Either (M, \mathcal{Q}) is a quaternionic manifold (if $n \geq 2$) or $(M, \mathcal{Q} = [g])$ is ant-self dual for $n = 1$.*
- 2). *There exists an integrable almost complex structure I_1^∇ on the twistor space Z which does not depend on the quaternionic connection ∇ .*

4. QUATERNIONIC KÄHLER MANIFOLDS WITH TORSION

An almost quaternionic Hermitian manifold (M, \mathcal{Q}, g) is called quaternionic Kähler with torsion (QKT) if there exists an almost quaternionic Hermitian connection $\nabla^T \in \Delta(\mathcal{Q})$ whose torsion tensor T is a 3-form which is $(1,2)+(2,1)$ with respect to each J_a [16], i.e. the tensor $T(X, Y, Z) := g(T(X, Y), Z)$ is totally skew-symmetric and satisfies the conditions

$$T(X, Y, Z) = T(J_a X, J_a Y, Z) + T(J_a X, Y, J_a Z) + T(X, J_a Y, J_a Z), \quad a = 1, 2, 3.$$

We recall that each QKT is a quaternionic manifold due to an observation made in [18]. The condition on the torsion implies that the $(0,2)$ -part of the torsion of a QKT connection vanishes. Applying Theorem 3.7, we obtain

Theorem 4.1. *Let $(M, \mathcal{Q}, \nabla^T)$ be a QKT and $\nabla^0 \in \Delta(\mathcal{Q})$ be a torsion-free quaternionic connection. Then the complex structure $I_1^{\nabla^T}$ on the twistor space Z constructed in [16] coincides with the complex structure $I_1^{\nabla^0}$ constructed in [24].*

REFERENCES

- [1] D.V.Alekseevsky, *Riemannian spaces with exceptional holonomy groups*, Funkcional. Anal. Prilozhen. **2** (1968), 1-10. 2
- [2] D.V.Alekseevsky, *Compact quaternion spaces*, Funkcional. Anal. Prilozhen. **2** (1968), 11-20. 2
- [3] B. Alexandrov, G. Grantcharov, S. Ivanov, *Curvature properties of twistor spaces of quaternionic Kähler manifolds*, Journal of Geometry **62** (1998), 1-12. 3, 4
- [4] D.V.Alekseevsky, S.Marchiafava, *Quaternionic structures on a manifold and subordinated structures*, Annali di Mat. Pura e Appl. (4) **171** (1996), 205-273. 4, 8, 9, 10
- [5] D.V.Alekseevsky, S.Marchiafava, M.Pontecorvo, *Compatible complex structures on almost quaternionic manifolds*, Trans. Amer. Math. Soc. **351** (1999), 997-1014. 7, 8, 10
- [6] M. F. Atiyah, N. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, **362**(1978), 425 – 461. 2, 5, 10
- [7] A.Besse, *Einstein manifolds*, Springer-Verlag, New York, 1987. 4
- [8] E.Bonan, *Sur les G -structures de type quaternionen*, Cahiers de Topologie et Géométrie Différentielle, **9** (1967), 389-461. 2
- [9] R. Bryant, *Lie groups and twistor spaces*, Duke Math. J., Vol. **52**, 1(1985), 223 – 261. 4

- [10] J. Davidov, O. Muškarov, *Twistor spaces with Hermitian Ricci tensor*, Proc. Amer. Math. Soc. **109**(1990), 1115 – 1120. 2, 5
- [11] J. Eells, S. Salamon, *Constructions twistorielles des applications harmoniques*, C. R. Acad. Sc. Paris, **296**(1983), 685 – 687. 2, 5
- [12] N. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. **43** (1981), 133–150. 2
- [13] S.J.Gates, C.M.Hull, M.Roëk, *Twisted multiplets and new supersymmetric non-linear σ -models*, Nucl. Phys. **B248**(1984), 157 – 186. 2
- [14] P.S.Howe, G.Papadopoulos, *Further remarks on the geometry of two dimensional nonlinear σ models*, Class. Quantum Grav. **5** (1988), 1647-1661. 2
- [15] P.S.Howe, G.Papadopoulos, *Twistor spaces for hyper-Kähler manifolds with torsion*, Phys. Lett. **B379**(1996), 81 – 86. 2, 5
- [16] P.S.Howe, A.Opfermann, G.Papadopoulos, *Twistor spaces for QKT manifolds*, Comm. Math. Phys. **197**(1998), 713 – 727.. 2, 5, 11
- [17] S.Ishihara, *Quaternion Kählerian manifolds*, J. Diff. Geom. **9**, (1974), 483-500. 2, 4
- [18] S. Ivanov, *Geometry of Quaternionic Kähler connections with torsion*, J. Geom. Phys. **41** (2002), 235-257. 11
- [19] S. Ivanov, I. Minchev, *Quaternionic Ka"hler and hyperKa"hler manifolds with torsion and twistor spaces*, J. Reine Angew. Math. **567** (2004), 215–233. 3, 4, 5, 10
- [20] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, 2 volumes, Interscience Publ., New York, 1963, 1969. 4, 5, 9
- [21] H.Nishino, *Alternative $N = (4, 0)$ superstring and σ - models*, Phys. Lett. **B 355** (1995), 117-126. 2
- [22] M.Obata, *Affine connections on manifolds with almost complex quaternionic or Hermitian structures*, Jap. J. Math. **26** (1956), 43-79. 2
- [23] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143-171. 2, 5
- [24] S. Salamon, *Differential geometry on quaternionic manifolds*, Ann. Sci. Ec. Norm. Sup. Ser (4) **19** (1986), 31-55. 2, 5, 10, 11
- [25] S.Salamon *Special structures on four manifold*, Riv. Math. Univ. Parma (4) **17** (1991), 109-123. 2
- [26] S.Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Math. Series, vol. 201 (Longman, New-York, 1989). 2
- [27] K. Sekigawa, *Almost Hermitian structures on twistor bundles*, Proc. Ramanujan Centennial Int. Conf., Annamalaiagar (1987), 127 – 136. 4

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